Degree of homogeneity and substitution rates

Given the following production function:

$$P(L, K) = \sqrt{L \cdot K^{\alpha}}, \quad \text{where } \alpha > 0$$

a) Prove that the function is homogeneous and establish for which values of α the function exhibits increasing returns to scale.

b) If $\alpha = 1$, plot the level curves for $P = 5$ and $P = 10$. Compute the substitution rates and provide an economic interpretation of the result and its relationship with the level curves.

Solution

a)

Given the production function:

$$P(L,K) = \sqrt{L \cdot K^\alpha}, \quad \text{where } \alpha > 0$$

To prove that the function is homogeneous, we analyze its behavior when multiplying both factors by a positive constant $t > 0$:

$$\begin{aligned} P(tL, tK) &= \sqrt{tL \cdot (tK)^\alpha} \\ &= \sqrt{tL \cdot t^\alpha K^\alpha} \\ &= \sqrt{t^{1+\alpha} L \cdot K^\alpha} \\ &= t^{\frac{1+\alpha}{2}} \sqrt{L \cdot K^\alpha} \\ &= t^{\frac{1+\alpha}{2}} P(L,K) \end{aligned}$$

This demonstrates that the function is homogeneous of degree:

$$k = \frac{1+\alpha}{2}$$

To determine when the function exhibits increasing returns to scale, we analyze the value of k:

- If $k > 1$, there are increasing returns to scale.
- If $k = 1$, there are constant returns to scale.
- If $k < 1$, there are decreasing returns to scale.

Thus:

$$\frac{1+\alpha}{2} > 1 \implies 1 + \alpha > 2 \implies \alpha > 1$$

The function is homogeneous of degree $k = \frac{1+\alpha}{2}$ and exhibits increasing returns to scale when $\alpha > 1$.

b)

With $\alpha = 1$, the production function simplifies to:

$$P(L,K) = \sqrt{L \cdot K}$$

The level curves (isoquants) for $P = c$ are obtained by solving:

$$\sqrt{L \cdot K} = c \implies L \cdot K = c^2$$

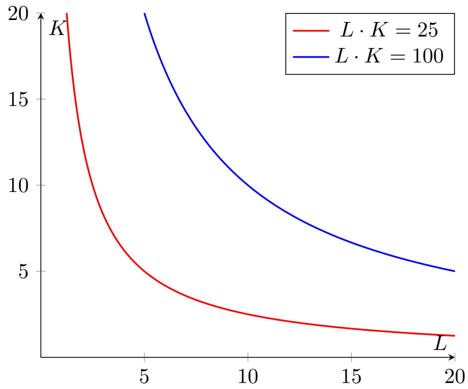
For $P = 5$:

$$L \cdot K = 25$$

For $P = 10$:

$$L \cdot K = 100$$

These equations represent hyperbolas in the (L, K) plane.
Graph of the level curves:



Computation of the marginal rates of technical substitution (MRTS):
The marginal rate of technical substitution between L and K is defined as:

$$\text{MRTS}_{LK} = -\frac{\partial K}{\partial L}\Big|_{P \text{ constant}} = -\frac{MP_L}{MP_K}$$

We compute the marginal products:

$$MP_L = \frac{\partial P}{\partial L} = \frac{1}{2}(L \cdot K)^{-1/2} \cdot K = \frac{K}{2\sqrt{L \cdot K}}$$
$$MP_K = \frac{\partial P}{\partial K} = \frac{1}{2}(L \cdot K)^{-1/2} \cdot L = \frac{L}{2\sqrt{L \cdot K}}$$

Thus, the MRTS is:

$$\text{MRTS}_{LK} = -\frac{MP_L}{MP_K} = -\frac{\frac{K}{2\sqrt{L \cdot K}}}{\frac{L}{2\sqrt{L \cdot K}}} = -\frac{K}{L}$$

Economic interpretation:

- **Negative MRTS:** The negative sign indicates that to increase L while keeping P constant, it is necessary to decrease K.
- **Dependence on $\frac{K}{L}$:** The MRTS depends on the ratio between K and L. As L increases and K decreases along an isoquant, the ratio $\frac{K}{L}$ decreases, reflecting a diminishing MRTS in absolute value.
- **Relation to level curves:** The isoquants are hyperbolas convex to the origin, indicating diminishing marginal rates of substitution between L and K. The shape of the isoquants and the MRTS consistent with $-\frac{K}{L}$ demonstrate that less reduction in K is required to offset additional increments in L while maintaining P constant.

Conclusion: The marginal rate of technical substitution $\text{MRTS}_{LK} = -\frac{K}{L}$ indicates how the firm can substitute capital for labor without altering the production level, reflecting the nature of the level curves and diminishing marginal rates of substitution in production.